Leonardo's Elevated Polyhedra - Models

Rinus Roelofs
Lansinkweg 28
7553AL Hengelo
The Netherlands
E-mail: rinus@rinusroelofs.nl
www.rinusroelofs.nl

Information

Rinus Roelofs was born in 1954. After studying Applied Mathematics at the Technical University of Enschede, he took a degree from the Enschede Art Academy with a specialization in sculpture. His commissions come largely from municipalities, institutions and companies in the Netherlands, but his work has been exhibited further afield, including in Rome as part of the Escher Centennial celebrations in 1998. From 2003 he is a yearly presenter at the Bridges conference, a conference on connections between art and mathematics.

Publications

Ecken und Kanten – Europaweit, 2014, catalogue exhibition Mathematikum Giessen, Germany.
The Discovery of a New Series of Uniform Polyhedra, 2013, Bridges Proceedings.
Connected Holes, 2008, catalogue exhibition Technical University of Enschede, Netherlands.

Introduction

The main subject of my art my fascination about mathematics. And to be more precise: my fascination about mathematical structures. Mathematical structures can be found all around us. We can see them everywhere in our daily live. The use of these structures as visual decoration is so common that we don't even see this as mathematics. But studying the properties of these structures and especially the relation between the different structures can bring up questions. Questions that can be the start of interesting artistic explorations.
Artistic explorations of this kind mostly leads to intriguing designs of sculptural objects, which are then made in all kind of materials, like paper, wood, metal, acrylic, etc.. It all starts with amazement, trying to understand what you see. Solving those questions often leads to new ideas, new designs.
Since I use the computer as my main sketchbook these ideas come to reality first as a picture on the screen. From there I can decide what the next step towards physical realization has to be. A rendered picture, an animation or a 3D physical model made by the use of CNC-milling, laser cutting or rapid prototyping. Most of the time the first physical model is a paper model, simply cut out, folded and glued together.
However many different techniques can be used nowadays, as well as many different materials. But it is all based on my fascination about mathematical structures.

In mathematics the field of Polyhedra deserves a special attention. It is nice to build real physical models from which you can learn a lot about the beauty of symmetry and structure. Once you have the first models of the Platonic solids you get inspired and you can come up with ideas about variations on these models. In the book "Divina Proportione" by Luca Pacioli and Leonardo da Vinci you will find drawings of the basic polyhedra and also an interesting variation, called elevation.

Paper models can be made quite easily. For the models described here we only need three different templates which can be found in the appendix.

In their book “La Divina Proportione” [2] Luca Pacioli and Leonardo da Vinci introduced the elevated versions of all of the Platonic polyhedra as well as of some of the Archimedean polyhedra. The Platonic solids as well as their elevations as they are drawn by Leonardo da Vinci are shown in Figure 1 and 2.

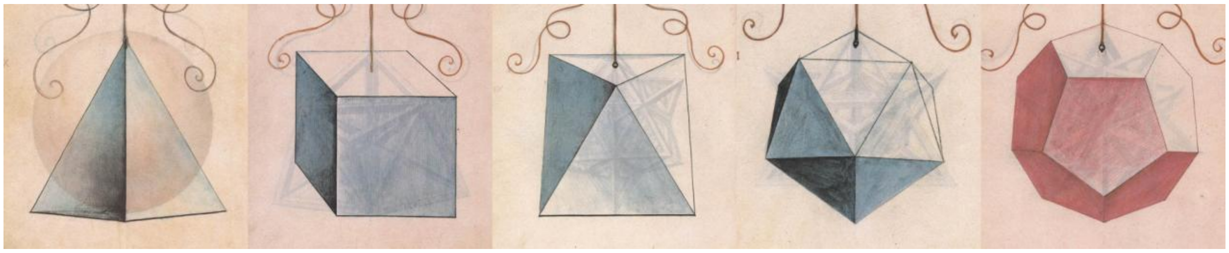

Figure 1: *Leonardo’s drawings of the Platonic solids.*

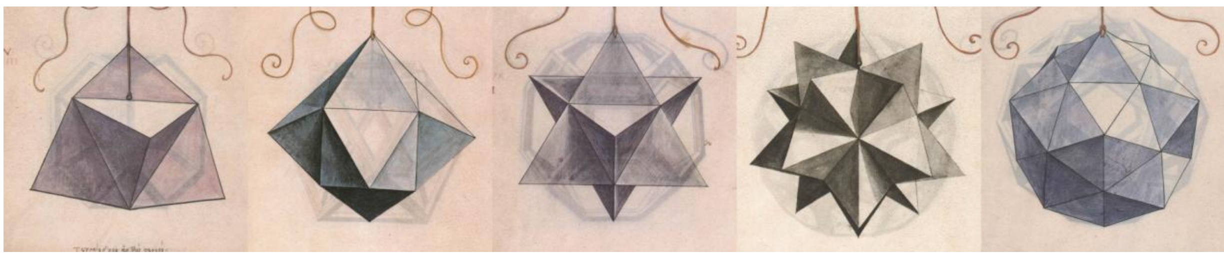

Figure 2: *Leonardo’s drawings of the elevations of the Platonic solids.*

What exactly is an elevated version of a polyhedron? In “La Divina Proportione” [3],chapter XLIX, paragraph XI.XII, Pacioli describes the elevated version of the cube as follows: “… it is enclosed by 24 triangular faces. This polyhedron is built out of 6 four-sided pyramids, together building the outside of the object as you can see it with your eyes. And there is also a cube inside, on which the pyramids are placed. But this cube can only be seen by imagination, because it is covered by the pyramids. The 6 square faces are the bottom faces of the 6 pyramids.”.
So in total this object is composed of 24 equilateral triangular faces plus 6 hidden square faces, as can be seen in the exploded views in Figure 3.

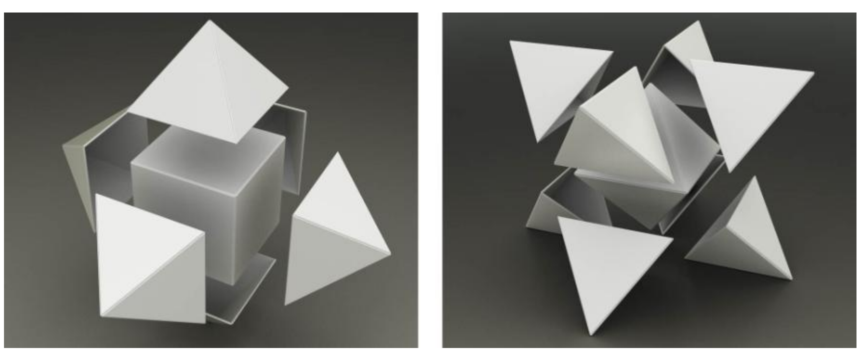

Figure 3: *Exploded views of the elevated cube and octahedron.*

About the “Octocedron Elevatus”, Pacioli writes (Chapter L, paragraph XIX.XX): “And this object is built with 8 three-sided pyramids, that can be seen with your eyes, and an octahedron inside, which you can only see by imagination.”. This means that the object is composed of 32 equilateral triangular faces of which 8 are hidden.

Pacioli describes the process of elevation as putting pyramids, built with equilateral triangles, on each of the faces of the polyhedra. The result of this operation is a double layered object and has much similarities with the stellated version of a polyhedron, a beautiful example of which can be seen in M.C. Escher's print "Gravity". The way Escher opened up the polyhedron shows us both layers very well. Or in Escher's own words: "This star-dodecahedron is built with twelve five pointed stars. On each of these platforms lives a monster without a tale and his body is captured under a five sided pyramid.". Escher is talking about pyramids placed on platforms, just like Pacioli's elevations.

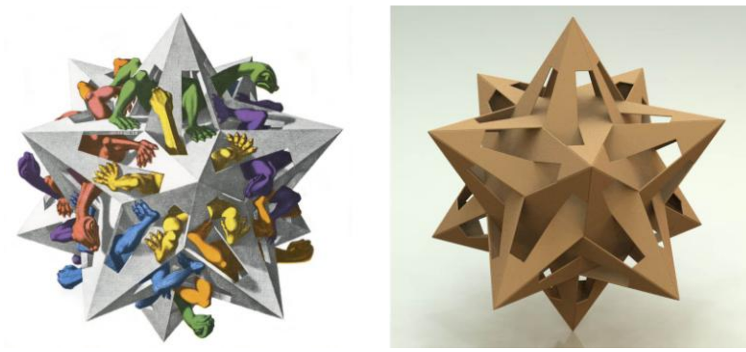

Figure 4: *M.C. Escher – "Gravity". Basic shape is the Stellated dodecahedron.*

The way Escher opened up the polyhedron turned out to be the perfect solution for making models of the elevated polyhedra.

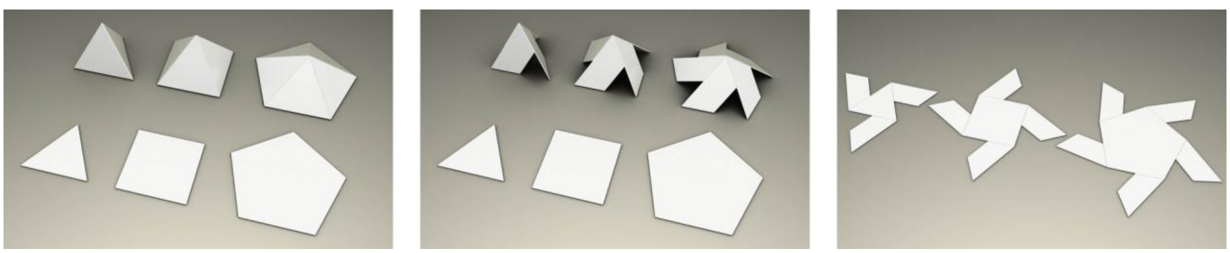

Figure 5: *Developing of the basic elements for the elevation models.*

We can construct the elements that we need to build the model of an elevated polyhedron as follows: connect the faces of the "opened" pyramid to the edges of its base. For the elevated Platonic solids we need to that with the three, four and five sided pyramids. So we end up with three different building elements (Figure 5, right) of which you can find the drawings in the appendix.

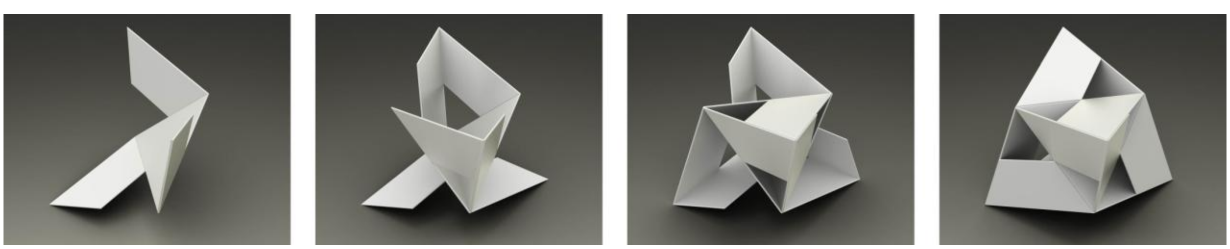

Figure 6: *Building the elevated tetrahedron.*

Figure 6a: *Building the first elevation of one of the faces of the tetrahedron.*

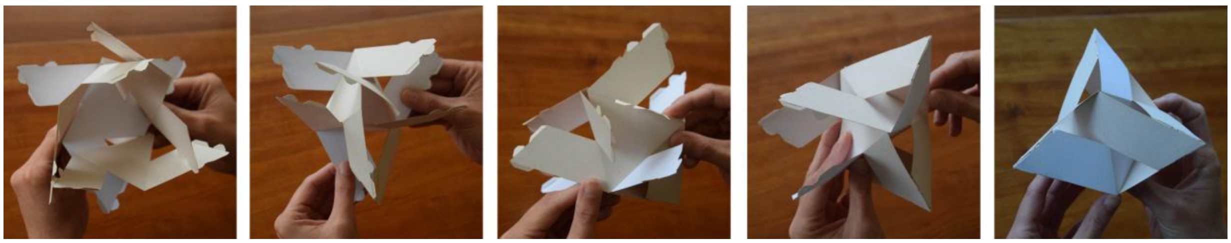

Figure 6b: *Connectiing all the parts after turning over the model.*

We start with four triangular elements to first build the model of the elevated tetrahedron. The building process can be described as follows: above each triangular face we have to build a triangular pyramid.

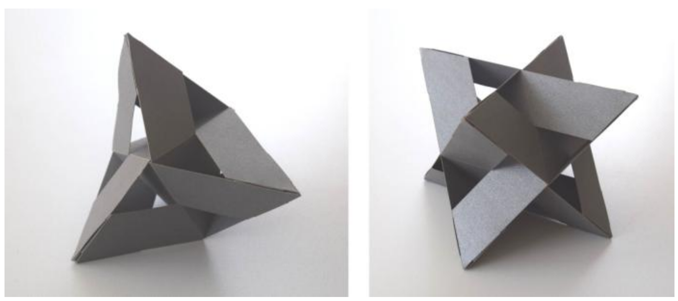

Figure 7: *Models of the elevated tetrahedron and elevated octahedron.*

And with eight of those elements we can build the model of the elevated octahedron.

Figure 8: *Building the elevated icosahedron.*

For the third model, the elevated icosahedron, we need to cut out twenty triangular elements.

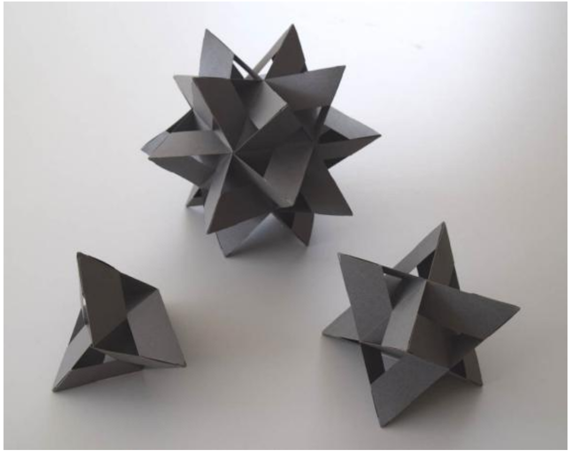

Figure 9: *Models of the elevated polyhedra with the triangular element.*

So now we have all the stellated Platonic solids which can be build with the triangular element.
For the next model, the elevated cube we need to cut out six square elements. And now we make square pyramids above each of the square faces of the cube.

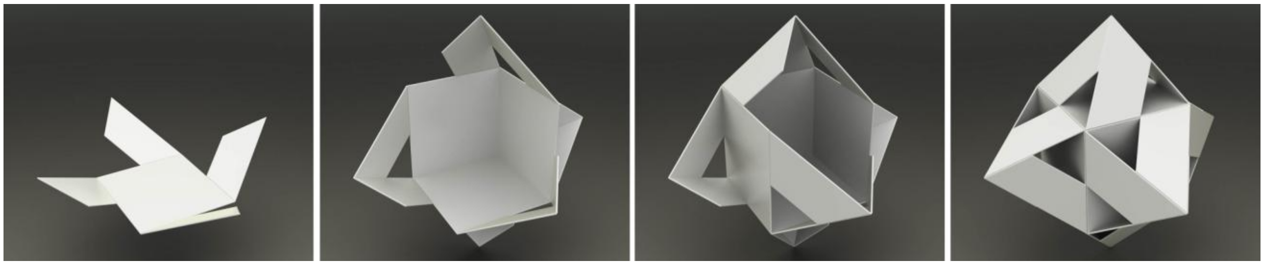

Figure 10: *Building the elevated cube*

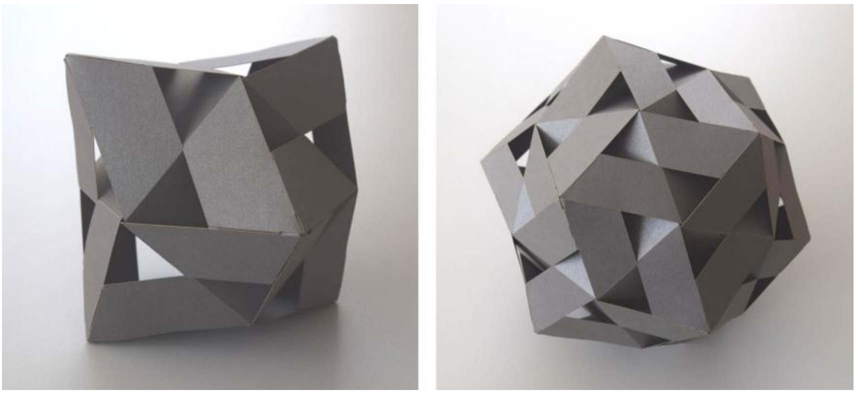

Figure 11: *Models of the elevated cube and elevated dodecahedron.*

The final model of the set of stellated platonic solids, the stellated dodecahedron, can be made with twelve pentagonal elements.

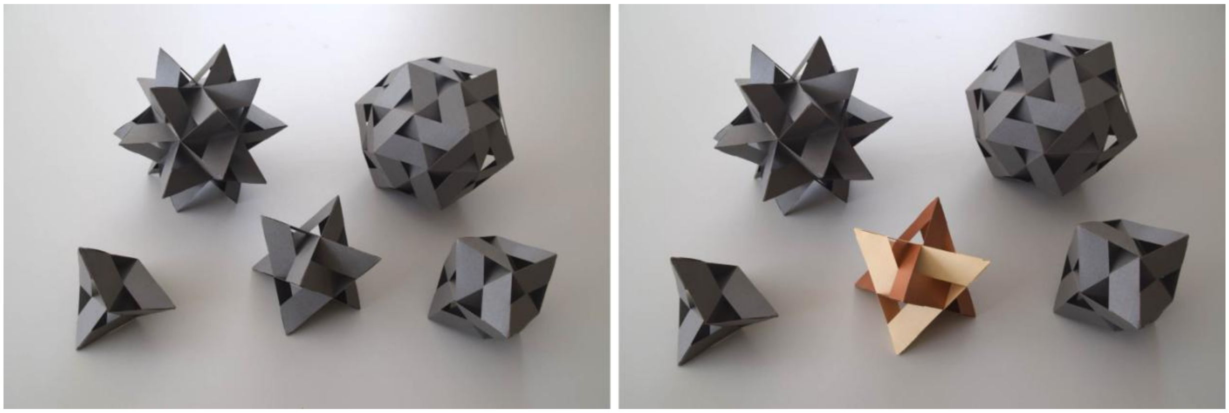

Figure 12: *Models of the elevated Platonic solids.*

Looking at the complete set of models we have built so far, we see that one of those models, the stellated octahedron, can be seen as a compound of two tetrahedra. Using two different colors makes this visible and there is a nice resemblance between the paper model and Escher's "Double Planetoid"

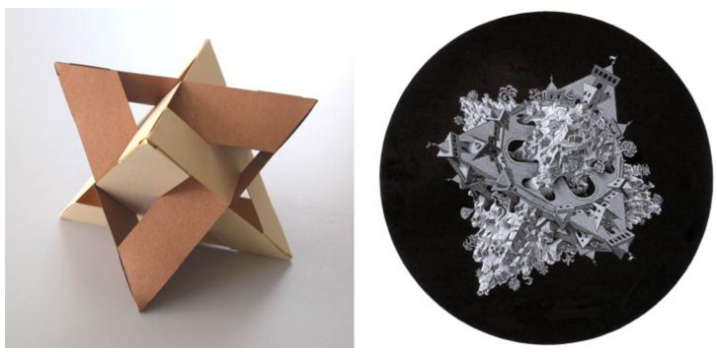

Figure 13: *The elevated octahedron is a compound of two tetrahedra.*

The next step is look for the possible stellated polyhedra we can get from the Archimedean solids.

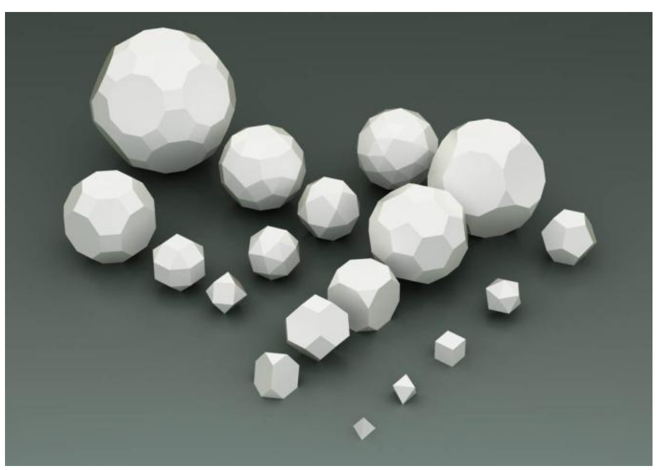

Figure 14: *The Archimedean and Platonic solids*

To make a stellation of a polyhedron we have to place a pyramid with equilateral triangular faces on each of the faces of the polyhedron. As you can see in Figure 15, only three, four and five sided pyramids are possible.

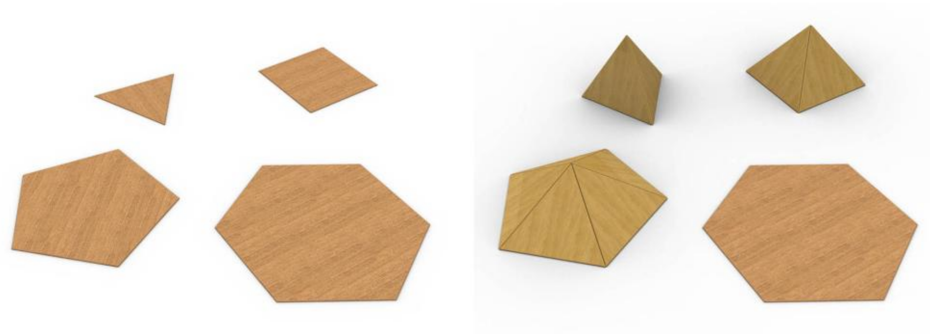

Figure 15: *Possible elevations of regular polygons.*

Of the total set of the Archimedean solids only six solids can be used for the stellation process. They are shown in Figure 16.

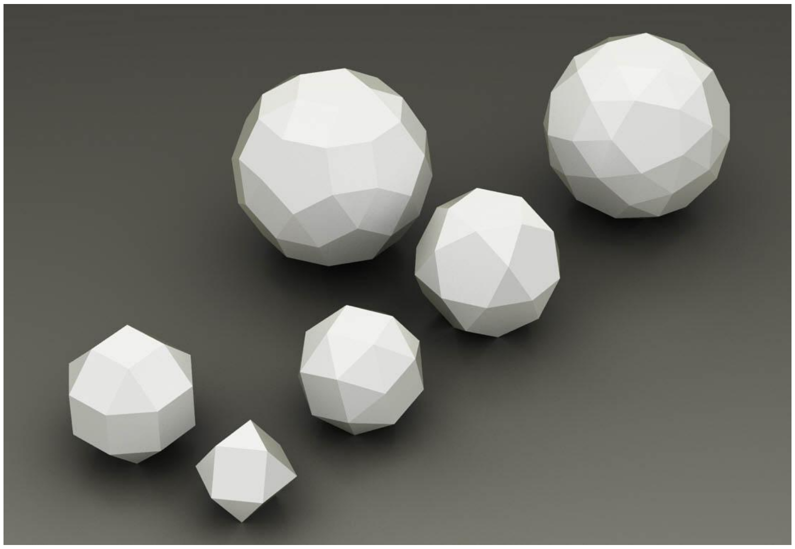

Figure 16: *The six Archimedean Solids which can be elevated.*

In the book "Divina Proportione" we find only three: the cuboctahedron, the icosidodecahedron and the rhombicuboctahedron.

Figure 17: *Leonardo's drawings of three Archimedean solids.*

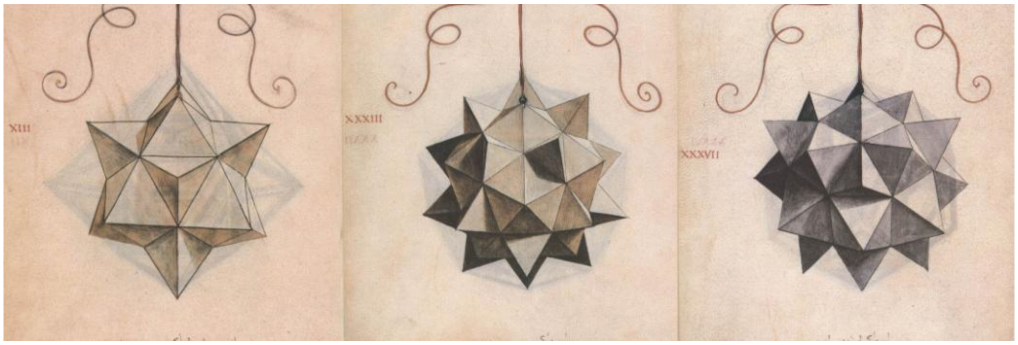

Figure 18: *Leonardo's drawings of the elevations of three Archimedean solids.*

We can make the paper models of all of them with the three different elements.

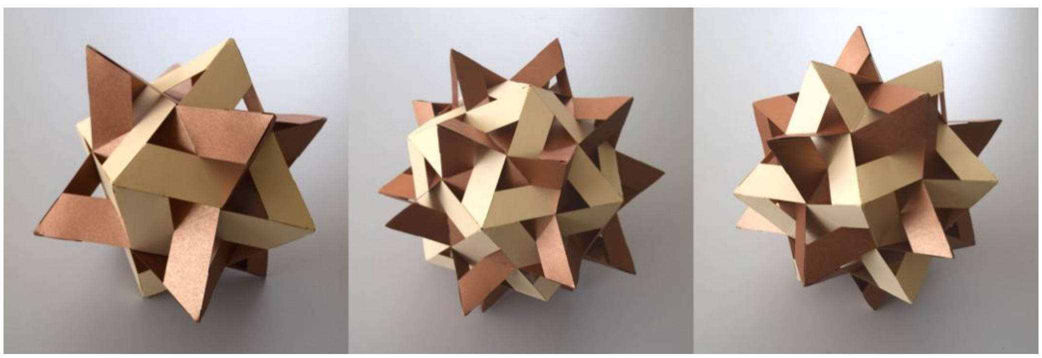

Figure 19: *Paper models of three of the Archimedean solids.*

And besides these three Archimedean elevations we can also build the stellated snubcube, the elevated snubdodecahedron and the elevated rhombicosidodecahedron. In total we now have eleven elevated polyhedra. There is an special subset of these eleven polyhedra with interesting properties. In Figure 20 the so called Ring Polyhedra are shown. These polyhedra can be colored with only two colors in such a way that no two adjacent faces have the same color. The elevated versions of these polyhedra will therefore be compounds and we can make them the models in two different colors.

Figure 20: *The five Ring Polyhedra.*

In Figure 21 the compounds of Leonardo's elevations are shown..

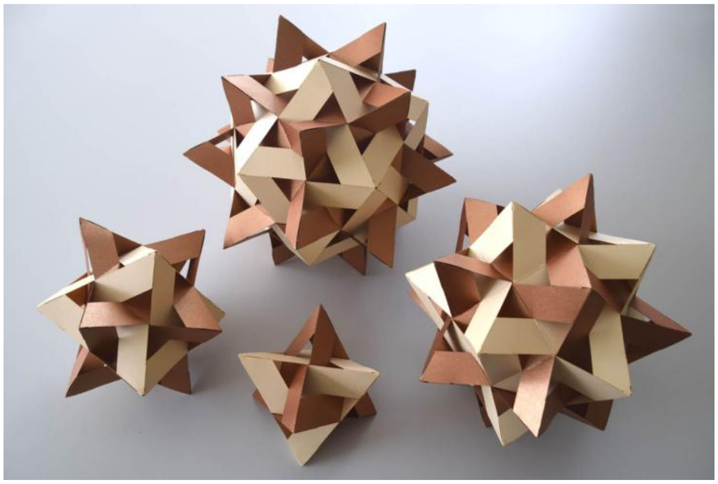

Figure 21: *Four of Leonardo's elevated polyhedra are compounds.*

There are many more polyhedra we can build with our elements. The first group we can study is the group of the deltahedra. There are eight convex deltahedra and with our elements we can build elevated versions of each of them.

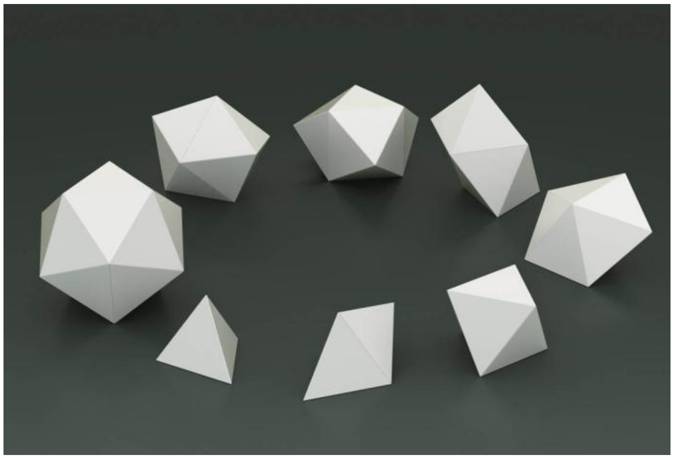

Figure 22: *The eight deltahedra.*

In fact from any polyhedron build with triangles, squares and/or pentagons a paper model of the elevated version can be build with our elements. A nice example is the Tetrahelix.

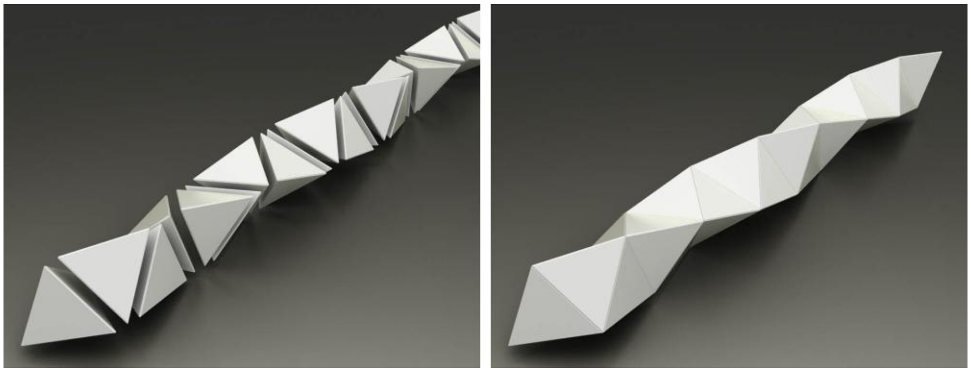

Figure 23: *Tetrahelix*

Especially when you use more than one color this will give you nice models.

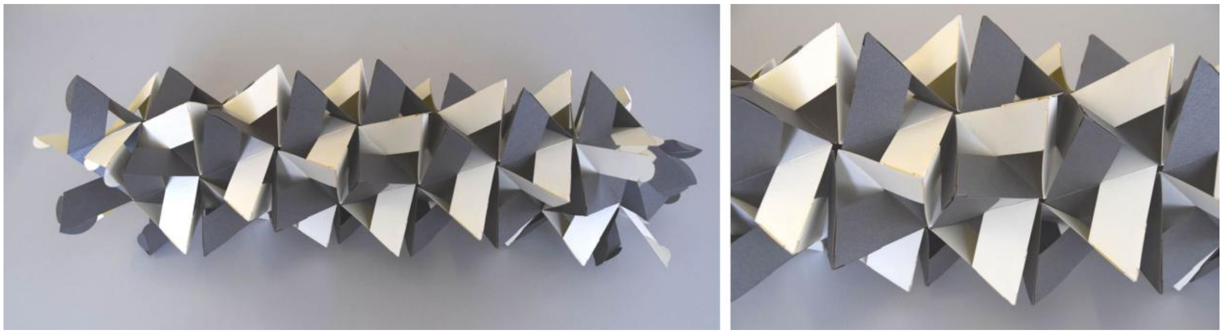

Figure 24: *Elevated version of the tetrahelix made with the triangular element.*

The idea of elevation cannot only be applied on polyhedra but also on flat tilings.

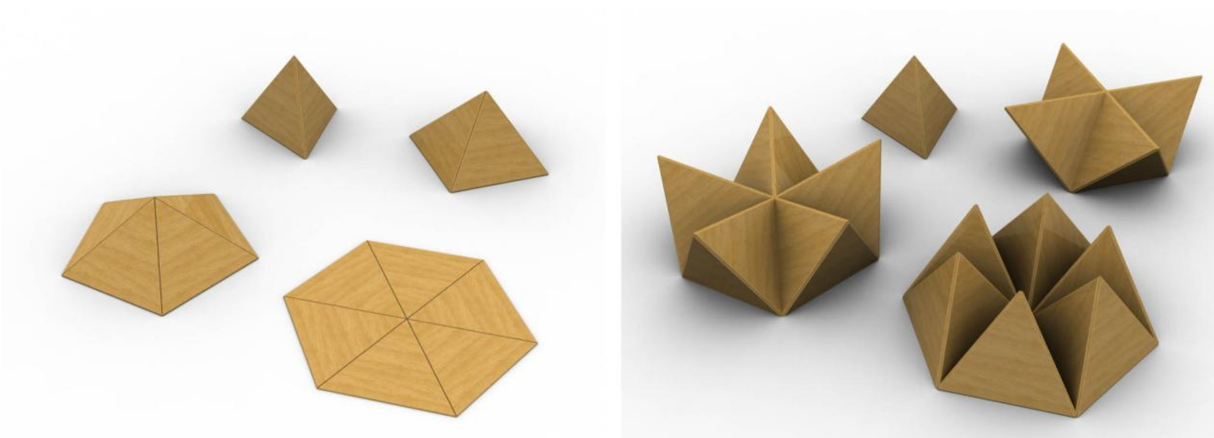

Figure 25: *Elevation is also possible with plane patterns.*

For example, starting with the tiling 3,3,3,3,3,3 in which all the tiles are equilateral tringles, we can put a triangular pyramid on each of the tiles. And when we open up the pyramids like Escher did we can make the model with the triangular elements as is shown in Figure 26.

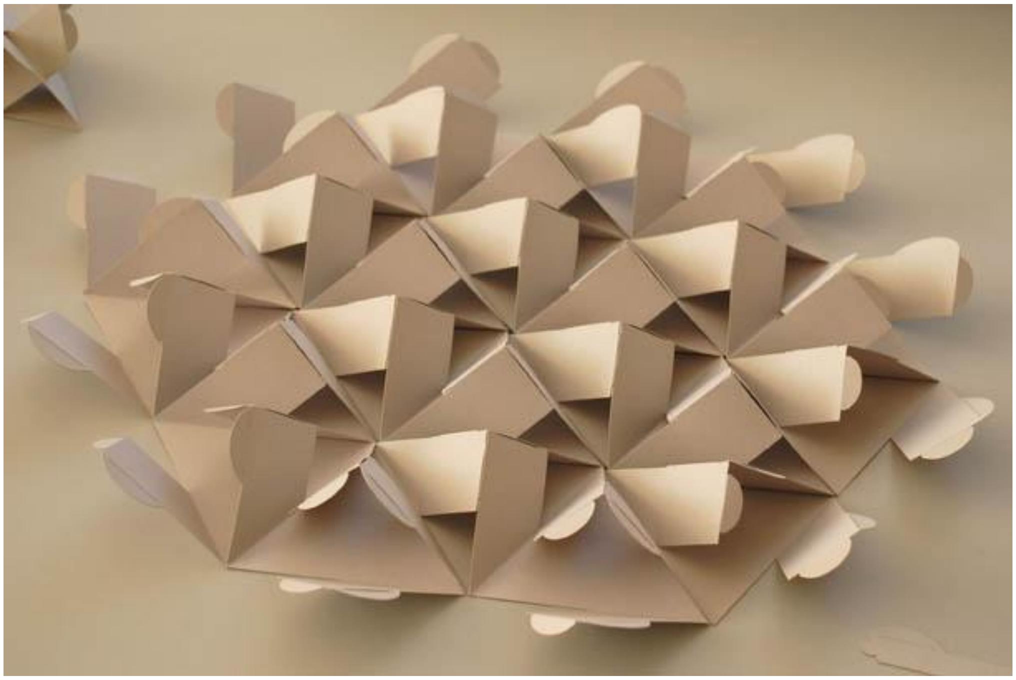

Figure 26: *Model of the elevation of the planar 3,3,3,3,3,3-tiling, built with the triangular element.*

We can apply the same strategy on the tiling with squares. This gives us two regular elevated tilings.

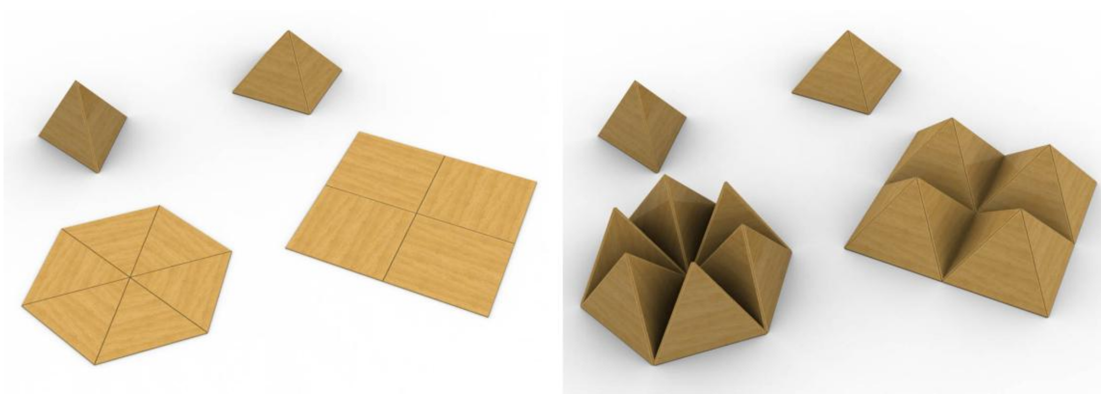

Figure 27: *Elevation of triangular and square pattern.*

There is one more Archimedean tiling that we can use in this way: tiling 3,3,4,3,4 shown in Figure 28.

Figure 28: *Elevation of the Archimedean pattern 3,3,4,3,4.*

The paper model of the elevation is shown in Figure 29.

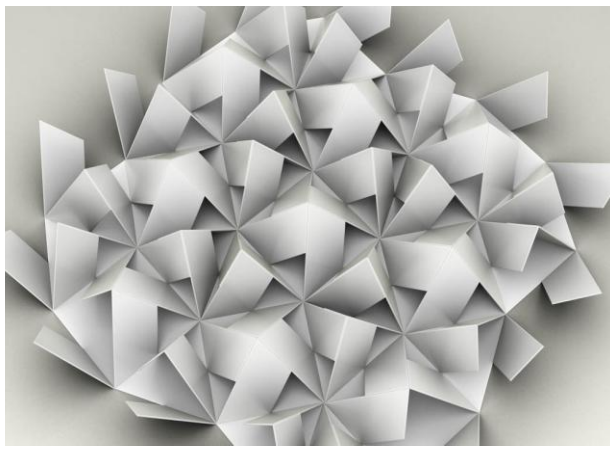

Figure 29: *Model built with the triangular and the square elements.*

But of course many variations can be made. So with only three different elements we can built a huge collection of interesting mathematical objects and structures.